11th Week Friday

5 Mayıs 2023 Cuma 10:29

Matrix Representations of Linear Transformations

$L : V \to W$ — $\dim(V) = m$, $\dim(W) = n$

$v \in V \mapsto L(v) = w \in W$

$A_{n\times m} \begin{bmatrix} | \\ v \\ | \end{bmatrix}_{m\times 1} = \begin{bmatrix} | \\ w \\ | \end{bmatrix}_{n\times 1}$

$L(v) = A v$ → Can we find such a matrix A which will represent L.

Ex. $L : \mathbb{R}^2 \to \mathbb{R}^3$ Find a matrix representation for L.

$(x,y) \mapsto (x+y, x-y, 0) \to (x+y)\begin{bmatrix}1\\0\\0\end{bmatrix}_{e_1} + (x-y)\begin{bmatrix}0\\1\\0\end{bmatrix}_{e_2} + 0\begin{bmatrix}0\\0\\1\end{bmatrix}_{e_3}$

$x\begin{bmatrix}1\\0\end{bmatrix}_{e_1} + y\begin{bmatrix}0\\1\end{bmatrix}_{e_2}$

$L((x,y)) = (x+y, x-y, 0)$, $(x,y) \in \mathbb{R}^2$, $(x+y, x-y, 0) \in \mathbb{R}^3$

$\rightsquigarrow \underset{?\ 3\times 2}{A} \underset{2\times 1 \text{ input}}{\begin{bmatrix} x \\ y \end{bmatrix}} = \underset{3\times 1 \text{ output}}{\begin{bmatrix} x+y \\ x-y \\ 0 \end{bmatrix}}$ $\quad A_{3\times 2}$

$A = \begin{bmatrix} | & | \\ L(e_1) & L(e_2) \\ | & | \end{bmatrix}_{3\times 2} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 0 \end{bmatrix}$

→ check! $\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ x-y \\ 0x+0y \end{bmatrix}$ ✓

! Input and output are given in the standard basis.

standard basis vectors of the input vector space $\Rightarrow \left\{ e_1 = \begin{bmatrix}1\\0\end{bmatrix}, e_2 = \begin{bmatrix}0\\1\end{bmatrix} \right\}$

$L(e_1)$ $\quad L(e_2)$

$L(e_1) = L((1,0)) = (1+0, 1-0, 0) = (1,1,0)$

$L(e_2) = L((0,1)) = (0+1, 0-1, 0) = (1,-1,0)$

Ex. $L : \mathbb{R}^3 \to \mathbb{R}^4$ Find a representation matrix for L.

$(x,y,z) \mapsto (x, x+z, 2x-y, y-3z)$

$e_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$ $e_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$ $e_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$

$L(e_1) = L((1,0,0)) = (1, 1+0, 2-0, 0-0)$

$L(e_2) = L((0,1,0)) = (0, 0+0, 2.0-1, 1-3.0)$

$L(e_3) = L((0,0,1)) = (0, 0+1, 2.0-0, 0-3.1)$

$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 2 & -1 & 0 \\ 0 & 1 & -3 \end{bmatrix}_{4\times 3} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$A_{B,C}$

$L : V \to W$ (V: standard form, W: standard form)

$B \to$ a basis of V $\quad B = \{b_1, b_2, \ldots, b_m\}$

$C \to$ a basis for W $\quad C = \{c_1, c_2, \ldots, c_n\}$

$A_{B,C} = \begin{bmatrix} | & | & & | \\ [L(b_1)]_C & [L(b_2)]_C & \cdots & [L(b_m)]_C \\ | & | & & | \end{bmatrix}_{n\times m}$

Ex. $L : \mathbb{R}^2 \to \mathbb{R}^3$ $\quad B = \{ [1], [-2] \}$

Ex/ $L: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ $B = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \end{bmatrix} \right\}$ ($\vec{b}_1$, $\vec{b}_2$)

$\rightarrow x\vec{b}_1 + y\vec{b}_2 \longmapsto (x+y, x-y, x)$ → in the standard basis

inputs in the form of B.

$L(b_1) = L\left(1\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 0\begin{bmatrix} -2 \\ 3 \end{bmatrix}\right) = (1+0, 1-0, 1)$

$L(b_2) = L\left(0\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1\begin{bmatrix} -2 \\ 3 \end{bmatrix}\right) = (0+1, 0-1, 0)$

$A_{B,E} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix}$

$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = x\begin{bmatrix} 1 \\ 2 \end{bmatrix} + y\begin{bmatrix} -2 \\ 3 \end{bmatrix}$ ($x \to 1$, $y \to 0$) $\begin{bmatrix} -2 \\ 3 \end{bmatrix} = x\begin{bmatrix} 1 \\ 2 \end{bmatrix} + y\begin{bmatrix} -2 \\ 3 \end{bmatrix}$ ($x \to 0$, $y \to 1$)

Ex/ $L: \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ $B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ $A_{B,E} =$

$(x, y) \quad (x+y, x-y, x)$

$L(b_1) \quad L(b_2)$